

Benders, metric and cutset inequalities for multicommodity capacitated network design

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Abstract Solving multicommodity capacitated network design problems is a hard task that requires the use of several strategies like relaxing some constraints and strengthening the model with valid inequalities. In this paper, we compare three sets of inequalities that have been widely used in this context: Benders, metric and cutset inequalities. We show that Benders inequalities associated to extreme rays are metric inequalities. We also show how to strengthen Benders inequalities associated to non-extreme rays to obtain metric inequalities. We show that cutset inequalities are Benders inequalities, but not necessarily metric inequalities. We give a necessary and sufficient condition for a cutset inequality to be a metric inequality. Computational experiments show the effectiveness of strengthening Benders and cutset inequalities to obtain metric inequalities.

Keywords Multicommodity capacitated network design · Benders decomposition · Metric inequalities · Cutset inequalities

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1 Introduction

Network design problems have received significant attention in the literature for more than two decades, as evidenced by numerous reviews [3, 12, 18, 19]. In particular, multicommodity capacitated network design problems have been widely studied [1, 2, 9, 15, 21].

A typical network design problem is defined over a directed network $G = (N, A, K)$ with node set N , arc set A , and commodity set K , each commodity k being represented by a triplet $(O(k), D(k), d_k)$, where $O(k)$ is the origin node, $D(k)$ is the destination node and $d_k > 0$ is the demand to be routed between $O(k)$ and $D(k)$. For each commodity k , we assume that there exists at least one path from $O(k)$ to $D(k)$ and we define the connected subnetwork $G^k = (N^k, A^k)$ that contains only nodes and arcs that belong to some path from $O(k)$ to $D(k)$. Given a nonconvex objective function, the problem is to find a minimum cost solution that satisfies the demands for all commodities, often subject to additional constraints, such as budgetary limits and topological restrictions.

In this context, many solution algorithms make use of the *multicommodity network flow subproblem*: given a capacity vector $\omega = (\omega_{ij})_{(i,j) \in A} \geq 0$, we wish to identify a multicommodity flow vector $x = (x_{ij}^k)_{(i,j) \in A^k}^k \geq 0$ satisfying

$$\sum_{j \in N_i^k(+)} x_{ij}^k - \sum_{j \in N_i^k(-)} x_{ji}^k = \begin{cases} d_k, & i = O(k), \\ 0, & i \neq O(k), D(k), \\ -d_k, & i = D(k), \end{cases} \quad \forall i \in N^k, \forall k \in K, \quad (1)$$

$$\sum_{k \in K} x_{ij}^k \leq \omega_{ij}, \quad \forall (i, j) \in A, \quad (2)$$

where $N_i^k(+) = \{j \in N^k | (i, j) \in A^k\}$ and $N_i^k(-) = \{j \in N^k | (j, i) \in A^k\}$. In this formulation, the value ω_{ij} represents whether or not a connection is established on arc (i, j) , and what is the capacity available to carry the demands through this arc. We say that ω is a *feasible* capacity vector if and only if there exists a multicommodity flow $x \geq 0$ satisfying (1) and (2).

For each commodity k , one of the flow conservation equations (1) is redundant. Hence, if we introduce dual variables $\pi = (\pi_i^k)_{i \in N^k}^k$ associated to these equations, we can assume $\pi_{O(k)}^k = 0$ for each $k \in K$. Introducing also dual variables $\alpha = (\alpha_{ij})_{(i,j) \in A} \geq 0$ associated to the capacity constraints (2), we have, by strong duality in linear programming, that ω is feasible if and only if the following dual problem, $D(\omega)$, is bounded:

$$Z(\omega) = \text{Maximize} \sum_{k \in K} d_k \pi_{D(k)}^k - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij} \quad (3)$$

$$\text{subject to} \quad \pi_j^k - \pi_i^k - \alpha_{ij} \leq 0, \quad \forall (i, j) \in A^k, \forall k \in K. \quad (4)$$

Note that the feasible domain for $D(\omega)$, P_D , is a full-dimensional polyhedral cone. Any solution in P_D is called a *ray* and we know, by Minkowski’s theorem, that every

such ray can be expressed as a nonnegative linear combination of the extreme rays of P_D (a ray is *extreme* if we cannot express it as the sum of other rays). We denote by $E(P_D)$ the set of extreme rays of P_D .

In this paper, we study the relationships between four well-known classes of inequalities characterizing the feasibility of a capacity vector ω :

- *Benders inequalities.* The dual problem $D(\omega)$ is bounded if and only if $Z(\omega) \leq 0$ if and only if

$$\sum_{k \in K} d_k \pi_{D(k)}^k - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij} \leq 0, \quad \forall (\pi, \alpha) \in P_D. \tag{5}$$

We call these the Benders inequalities, since they are used as cuts in the famous Benders decomposition algorithm [5]. In case $\alpha = 0$, we obtain a so-called *trivial* Benders inequality.

- *Benders inequalities associated to extreme rays.* Since any ray can be expressed as a nonnegative linear combination of the extreme rays of P_D , we can restrict Benders inequalities to the set of extreme rays and yet obtain a characterization of the feasibility of ω :

$$\sum_{k \in K} d_k \pi_{D(k)}^k - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij} \leq 0, \quad \forall (\pi, \alpha) \in E(P_D). \tag{6}$$

- *Metric inequalities.* The dual problem, $D(\omega)$, is bounded if and only if, for any $\alpha \geq 0$, we have

$$\sum_{(i,j) \in A} \omega_{ij} \alpha_{ij} \geq \max \left\{ \sum_{k \in K} d_k \pi_{D(k)}^k \mid \pi_j^k - \pi_i^k \leq \alpha_{ij}, \forall (i,j) \in A^k, \forall k \in K \right\}. \tag{7}$$

The problem on the right-hand side of this inequality is the dual of a shortest path problem, with arc distances equal to $\alpha \geq 0$. Assume we solve this problem and obtain the shortest path lengths $\pi(\alpha) = (\pi_i^k(\alpha))_{i \in N^k}^{k \in K}$ between $O(k)$ and i , for each node $i \in N^k$ and commodity $k \in K$. We then have the following metric inequalities:

$$\sum_{k \in K} d_k \pi_{D(k)}^k(\alpha) - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij} \leq 0, \quad \forall \alpha \geq 0. \tag{8}$$

Note that $(\pi(\alpha), \alpha) \in P_D$; hence, metric inequalities are a special case of Benders inequalities.

- *Cutset inequalities.* Let $S \subset N$ be a non-empty subset of N and $\bar{S} = N \setminus S$ its complement; (S, \bar{S}) is a cutset, i.e., the set of arcs that connect a node in S to a node in \bar{S} . Also, let $K(S, \bar{S}) \subseteq K$ be the set of commodities having their origin in S and their destination in \bar{S} and $d_{(S, \bar{S})} = \sum_{k \in K(S, \bar{S})} d^k$. We then define the following cutset inequalities:

$$\sum_{(i,j) \in (S, \bar{S})} \omega_{ij} \geq d_{(S, \bar{S})}, \quad \forall S \subset N, S \neq \emptyset. \tag{9}$$

These inequalities simply mean that there should be enough capacity on the arcs of any cutset to satisfy the demands that must be routed through that cutset. They are often used in branch-and-cut algorithms for multicommodity capacitated network design problems [4, 6].

Clearly, cutset inequalities are necessary for a capacity vector ω to be feasible, but it is well-known that they are not sufficient in general (they are in the single-commodity case, due to the max flow-min cut theorem). As we have just seen, Benders inequalities, Benders inequalities associated to extreme rays, and metric inequalities are both necessary and sufficient to characterize a feasible capacity vector ω . In this sense, the three classes of inequalities are equivalent. However, when it comes to use these inequalities in a solution algorithm, the question of which class to use is not necessarily a trivial one, because even if one class of inequalities is a subclass of another, it might be easier to generate inequalities in the larger class. For example, in a Benders decomposition algorithm, generating only Benders cuts associated to extreme rays is not always better than using other Benders inequalities, as suggested by our computational results, presented in Sect. 4.

To compare inequalities, we use the standard *dominance* criterion. Given a polyhedron $P = \{u \mid Au \leq b\}$ and two valid inequalities $\lambda^1 u \leq \lambda_0^1$ and $\lambda^2 u \leq \lambda_0^2$ for P , we say that the first dominates the second if $\lambda^2 u \leq \lambda^1 u, \forall u \in P$ and $\lambda_0^1 \leq \lambda_0^2$. Furthermore, if there exists $u \in P$ such that $\lambda^2 u < \lambda^1 u$, the dominance is strict.

In this paper, we show the following properties:

1. Every Benders inequality associated to an extreme ray is a metric inequality.
2. For a given $\alpha \geq 0$, the corresponding metric inequality dominates all Benders inequalities associated to that particular α ; as a consequence, any Benders inequality can be *strengthened* to a metric inequality by solving the shortest path problem in (7).
3. Cutset inequalities are a subclass of Benders inequalities, but are not necessarily metric inequalities (contrary to a common belief).
4. We give a necessary and sufficient condition for a cutset inequality to be a metric inequality.

Our computational experiments, performed on instances of the multicommodity capacitated fixed-charge network design problem, illustrate the following results:

1. On many instances, we show that a large proportion of cutset inequalities can be strengthened to metric inequalities.
2. For a standard Benders decomposition algorithm, we show that restricting the method to Benders inequalities associated to extreme rays is not necessarily more efficient than generating cuts associated to any ray, extreme or not (contrary to a common belief).
3. For the same algorithm, we show, however, that strengthening Benders cuts to generate only metric inequalities is computationally more efficient than using Benders inequalities generated by a linear programming solver.

This paper is organized as follows. In Sect. 2, we present our results concerning metric inequalities, comparing them to Benders inequalities. Section 3 is dedicated to our developments on cutset inequalities. Section 4 presents our computational results, while Sect. 5 ends this paper with some conclusions.

2 Benders and metric inequalities

In this section, we show two results that allow us to compare Benders and metric inequalities, and we illustrate these results on a simple example.

We already know that Benders inequalities associated to extreme rays is a subclass of Benders inequalities. Our first result states that Benders inequalities associated to extreme rays is also a subclass of metric inequalities. To prove this result, we use the following simple lemma.

Lemma 1 *If (π, α) is a ray of P_D , then $\pi_h^k \leq \pi_h^k(\alpha), \forall h \in N^k, \forall k \in K$.*

Proof Let $k \in K$ and P_h^k a shortest path (with respect to α) from $O(k)$ to $h \in N^k$. We then have $\pi_h^k = \sum_{(i,j) \in P_h^k} (\pi_j^k - \pi_i^k) \leq \sum_{(i,j) \in P_h^k} \alpha_{ij} = \pi_h^k(\alpha)$. \square

Proposition 1 *Any non-trivial Benders inequality associated to an extreme ray is a metric inequality.*

Proof Let (π, α) an extreme ray of P_D associated to a non-trivial Benders inequality. Assume that this inequality is not a metric inequality. This implies that there exists some commodity $l \in K$ such that $\pi_{D(l)}^l \neq \pi_{D(l)}^l(\alpha)$. By Lemma 1, we must have $\pi_{D(l)}^l < \pi_{D(l)}^l(\alpha)$. This implies that, on every path P^l from $O(l)$ to $D(l)$, there exists an arc (i, j) such that $\pi_j^l - \pi_i^l - \alpha_{ij} < 0$, called a *non-binding arc*; otherwise, we would have a path Q^l such that $\pi_j^l - \pi_i^l - \alpha_{ij} = 0$ on every arc $(i, j) \in Q^l$, and $\pi_{D(l)}^l = \sum_{(i,j) \in Q^l} (\pi_j^l - \pi_i^l) = \sum_{(i,j) \in Q^l} \alpha_{ij} \geq \pi_{D(l)}^l(\alpha)$, a contradiction.

Using this property, we now construct a set $T^l \subset N^l$ as follows: we visit each path P^l from $O(l)$ to $D(l)$, starting from $O(l)$, until we encounter the first non-binding arc (i, j) ; we then add to T^l (initialized to \emptyset) all nodes on P^l from j to $D(l)$. Clearly, $O(l) \notin T^l$ and $D(l) \in T^l$, i.e., T^l defines a cutset. Moreover, it is not possible to have an arc (i, j) such that $i \in T^l$ and $j \notin T^l$. Indeed, $i \in T^l$ implies that there is a path from $O(l)$ to i containing a non-binding arc and $j \in N^l$ implies that there is a path from j to $D(l)$. Connecting these two paths through arc (i, j) , we would obtain a path from $O(l)$ to $D(l)$ containing a non-binding arc preceding node j . Hence, if $i \in T^l$, we must have $j \in T^l$.

Let $\delta = \min_{i \notin T^l, j \in T^l} \{-\pi_j^l + \pi_i^l + \alpha_{ij}\} > 0$, and define $(\bar{\pi}, \alpha)$ and $(\tilde{\pi}, 0)$ as follows:

$$\begin{aligned} \bar{\pi} &= \pi, \text{ except for } \bar{\pi}_i^l = \pi_i^l + \delta, \forall i \in T^l; \\ \tilde{\pi} &= 0, \text{ except for } \tilde{\pi}_i^l = -\delta, \forall i \in T^l. \end{aligned}$$

We then have:

- (1) $(\bar{\pi}, \alpha)$ is a ray, since for any $(i, j) \in A^l$, there are three possible cases:
 - (a) $i \in T^l$ and $j \in T^l$: $\bar{\pi}_j^l - \bar{\pi}_i^l - \alpha_{ij} = (\pi_j^l + \delta) - (\pi_i^l + \delta) - \alpha_{ij} = \pi_j^l - \pi_i^l - \alpha_{ij} \leq 0$;
 - (b) $i \notin T^l$ and $j \notin T^l$: $\bar{\pi}_j^l - \bar{\pi}_i^l - \alpha_{ij} = \pi_j^l - \pi_i^l - \alpha_{ij} \leq 0$;

- (c) $i \notin T^l$ and $j \in T^l$: $\bar{\pi}_j^l - \bar{\pi}_i^l - \alpha_{ij} = (\pi_j^l + \delta) - \pi_i^l - \alpha_{ij} \leq \pi_j^l + (-\pi_j^l + \pi_i^l + \alpha_{ij}) - \pi_i^l - \alpha_{ij} = 0$.
- (2) $(\tilde{\pi}, 0)$ is a ray, since for any $(i, j) \in A^l$, there are three possible cases:
 - (a) $i \in T^l$ and $j \in T^l$: $\tilde{\pi}_j^l - \tilde{\pi}_i^l = -\delta - (-\delta) = 0$;
 - (b) $i \notin T^l$ and $j \notin T^l$: $\tilde{\pi}_j^l - \tilde{\pi}_i^l = 0 - 0 = 0$;
 - (c) $i \notin T^l$ and $j \in T^l$: $\tilde{\pi}_j^l - \tilde{\pi}_i^l = -\delta - 0 < 0$.
- (3) $(\pi, \alpha) = (\bar{\pi}, \alpha) + (\tilde{\pi}, 0)$, which contradicts the hypothesis that (π, α) is an extreme ray. □

Even though Benders inequalities associated to extreme rays subsume the two other classes of inequalities, some solution algorithms might generate Benders inequalities that are not necessarily associated to extreme rays. If we let (π, α) a ray (not necessarily extreme) associated to a Benders inequality, one might solve a shortest path problem with respect to α and derive a metric inequality associated to the ray $(\pi(\alpha), \alpha)$. Our next result states that we then obtain a metric inequality that always dominates the original Benders inequality.

Proposition 2 Any Benders inequality associated to ray (π, α) is dominated by the metric inequality associated to ray $(\pi(\alpha), \alpha)$. The dominance is strict if and only if there exists $l \in K$ such that $\pi_{D(l)}^l < \pi_{D(l)}^l(\alpha)$.

Proof Using the definition of dominance given in the introduction, it suffices to show that

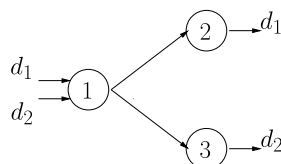
$$\sum_{k \in K} d_k \pi_{D(k)}^k - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij} \leq \sum_{k \in K} d_k \pi_{D(k)}^k(\alpha) - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij}.$$

But, this is immediate since, for each $k \in K$, $d^k > 0$ and $\pi_{D(k)}^k \leq \pi_{D(k)}^k(\alpha)$, by Lemma 1. It also follows immediately that the dominance is strict if and only if there exists $l \in K$ such that $\pi_{D(l)}^l < \pi_{D(l)}^l(\alpha)$. □

Example 1 Figure 1 shows a network with three vertices, two arcs and two commodities, with demands d_1 and d_2 , having their origins at node 1 and their destinations at nodes 2 and 3, respectively. Once we fix to 0 the dual variables associated to redundant constraints of the multicommodity flow subproblem (i.e., flow conservation equations for commodity 1, nodes 1 and 3, and for commodity 2, nodes 1 and 2) we obtain the following full-dimensional dual polyhedron:

$$P_D = \{(\pi, \alpha) = ((\pi_2^1, \pi_3^2), (\alpha_{12}, \alpha_{13})) \mid \pi_2^1 - \alpha_{12} \leq 0, \pi_3^2 - \alpha_{13} \leq 0, \alpha_{12}, \alpha_{13} \geq 0\}.$$

Fig. 1 Network of Example 1



We can easily enumerate the extreme rays of P_D :

$$(\pi(1), \alpha(1)) = ((1, 0), (1, 0));$$

$$(\pi(2), \alpha(2)) = ((0, 1), (0, 1));$$

$$(\pi(3), \alpha(3)) = ((-1, 0), (0, 0));$$

$$(\pi(4), \alpha(4)) = ((0, -1), (0, 0)).$$

There are two non-trivial Benders inequalities associated to these extreme rays:

$$\omega_{12} \geq d_1 \quad (\text{associated to } (\pi(1), \alpha(1)));$$

$$\omega_{13} \geq d_2 \quad (\text{associated to } (\pi(2), \alpha(2))).$$

If we take $\alpha = (1, 1)$, we obtain the metric inequality $\omega_{12} + \omega_{13} \geq d_1 + d_2$, associated to the ray $((1, 1), (1, 1)) = (\pi(1), \alpha(1)) + (\pi(2), \alpha(2))$, i.e., this metric inequality is obtained by aggregation of the two non-trivial Benders inequalities associated to extreme rays. In other words, the combination of the two Benders inequalities associated to extreme rays dominates this metric inequality, but none of them, individually, dominates it. Note that there are other Benders inequalities associated to $\alpha = (1, 1)$, for instance $\omega_{12} + \omega_{13} \geq d_1$ (associated to the ray $((1, 0), (1, 1))$) and $\omega_{12} + \omega_{13} \geq d_2$ (associated to the ray $((0, 1), (1, 1))$), which are not metric inequalities and are dominated by $\omega_{12} + \omega_{13} \geq d_1 + d_2$.

Suppose we perform two iterative solution algorithms. The first one generates, at each iteration, one Benders inequality associated to an extreme ray. The second algorithm generates, at each iteration, one Benders inequality and strengthens it to a metric inequality by solving a shortest path problem with α fixed at its value in the generated Benders inequality. Assume that, at some iteration, both algorithms solve the multicommodity flow subproblem with $\bar{\omega}$ satisfying $\bar{\omega}_{12} < d_1$ and $\bar{\omega}_{13} < d_2$. Also assume that $d_2 - \bar{\omega}_{13} > d_1 - \bar{\omega}_{12}$. Suppose the first algorithm generates the Benders inequality $\omega_{13} \geq d_2$. The second algorithm first generates the Benders inequality $\omega_{12} + \omega_{13} \geq d_2$ and then strengthens it to the metric inequality $\omega_{12} + \omega_{13} \geq d_1 + d_2$. This inequality is violated by $\bar{\omega}$ by the quantity $(d_2 - \bar{\omega}_{13}) + (d_1 - \bar{\omega}_{12}) > d_2 - \bar{\omega}_{13}$, the amount by which the Benders inequality $\omega_{13} \geq d_2$ is violated.

This example shows that, even if we select the best Benders inequality associated to an extreme ray, we might obtain a better metric inequality associated to a non-extreme ray. Thus, even though we can restrict ourselves to Benders inequalities associated to extreme rays, it might be interesting from a computational point of view to generate other Benders inequalities, in particular metric inequalities. Our computational results, presented in Sect. 4, confirm this observation.

A final remark is that one could certainly try to generate metric inequalities directly instead of generating Benders inequalities, then strengthening them to obtain metric inequalities. Indeed, [9] have proposed a cutting plane algorithm based on metric inequalities for a multicommodity capacitated network design problem. In their algorithm, a heuristic method is used to generate violated metric inequalities. In general, however, it is harder to derive violated metric inequalities than to obtain

violated Benders inequalities, in which case a state-of-the-art linear programming software package can be used to solve the multicommodity flow subproblem.

3 Cutset inequalities

In this section, we investigate the relationships between cutset inequalities and the three other classes of inequalities defined in the introduction. We first show that cutset inequalities are a subclass of Benders inequalities. To this purpose, we need to introduce, for any $S \subset N, S \neq \emptyset$, the ray $(\pi(S), \alpha(S))$ defined as follows:

$$\pi_i^k(S) = \begin{cases} 1, & \text{if } i \in \bar{S}, O(k) \in S \text{ and } D(k) \in \bar{S}, \forall i \in N^k, \forall k \in K, \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_{ij}(S) = \begin{cases} 1, & \text{if } i \in S \text{ and } j \in \bar{S}, \forall (i, j) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 3 *For any $S \subset N, S \neq \emptyset$, the cutset inequality associated to S corresponds to the Benders inequality associated to the ray $(\pi(S), \alpha(S))$.*

Proof By definition of $(\pi(S), \alpha(S))$, we have $\pi_{D(k)}^k(S) = 1$ for each commodity $k \in K(S, \bar{S})$ and $\alpha_{ij}(S) = 1$ if and only if $(i, j) \in (S, \bar{S})$ for each arc (i, j) , which implies:

$$\begin{aligned} \sum_{k \in K} d_k \pi_{D(k)}^k(S) - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij}(S) &= \sum_{k \in K(S, \bar{S})} d_k - \sum_{(i,j) \in (S, \bar{S})} \omega_{ij} \\ &= d_{(S, \bar{S})} - \sum_{(i,j) \in (S, \bar{S})} \omega_{ij}. \end{aligned}$$

The Benders inequality $\sum_{k \in K} d_k \pi_{D(k)}^k(S) - \sum_{(i,j) \in A} \omega_{ij} \alpha_{ij}(S) \leq 0$ is thus equivalent to the cutset inequality $d_{(S, \bar{S})} - \sum_{(i,j) \in (S, \bar{S})} \omega_{ij} \leq 0$. □

In general, a cutset inequality is not necessarily a metric inequality (consequently, it is not necessarily a Benders inequality associated to an extreme ray). The next proposition gives a necessary and sufficient condition for a cutset inequality to be a metric inequality.

Proposition 4 *For any $S \subset N, S \neq \emptyset$, the cutset inequality associated to S is a metric inequality if and only if for each $k \in K$*

- (a) *if $O(k) \in S$ and $D(k) \in \bar{S}$, there exists a path from $O(k)$ to $D(k)$ that crosses (S, \bar{S}) only once;*
- (b) *if $O(k) \in \bar{S}$ or $D(k) \in S$, there exists a path from $O(k)$ to $D(k)$ that never crosses (S, \bar{S}) .*

Proof Assume we solve the shortest path problem associated to the distances $\alpha(S)$ and obtain the shortest path lengths $\pi(\alpha(S))$. By definition of $\alpha(S)$, for each $k \in K$, there are three possible outcomes:

- (1) $\pi_{D(k)}^k(\alpha(S)) = 0$, which means that there exists a path from $O(k)$ to $D(k)$ that never crosses (S, \bar{S}) ;
- (2) $\pi_{D(k)}^k(\alpha(S)) = 1$, which means that there exists a path from $O(k)$ to $D(k)$ that crosses (S, \bar{S}) only once;
- (3) $\pi_{D(k)}^k(\alpha(S)) > 1$, which means that there is no path from $O(k)$ to $D(k)$ that never crosses (S, \bar{S}) or that crosses (S, \bar{S}) only once.

Now, let us assume that the cutset inequality associated to $S \subset N, S \neq \emptyset$, is a metric inequality. For any $k \in K$, we then have $\pi_{D(k)}^k(\alpha(S)) = \pi_{D(k)}^k(S)$, and

- (a) if $O(k) \in S$ and $D(k) \in \bar{S}$, then, by definition of $\pi(S)$, we have $1 = \pi_{D(k)}^k(S) = \pi_{D(k)}^k(\alpha(S))$, which implies that there exists a path from $O(k)$ to $D(k)$ that crosses (S, \bar{S}) only once;
- (b) if $O(k) \in \bar{S}$ or $D(k) \in S$, then, by definition of $\pi(S)$, we have $0 = \pi_{D(k)}^k(S) = \pi_{D(k)}^k(\alpha(S))$, which implies that there exists a path from $O(k)$ to $D(k)$ that never crosses (S, \bar{S}) .

Conversely, let us assume that, for each $k \in K$, we have either

- (a) $O(k) \in S$ and $D(k) \in \bar{S}$, in which case there exists a path from $O(k)$ to $D(k)$ that crosses (S, \bar{S}) only once, which implies that $\pi_{D(k)}^k(\alpha(S)) = 1 = \pi_{D(k)}^k(S)$, by definition of $\pi(S)$; or
- (b) $O(k) \in \bar{S}$ or $D(k) \in S$, in which case there exists a path from $O(k)$ to $D(k)$ that never crosses (S, \bar{S}) , which implies $\pi_{D(k)}^k(\alpha(S)) = 0 = \pi_{D(k)}^k(S)$, by definition of $\pi(S)$.

Thus, for each $k \in K, \pi_{D(k)}^k(\alpha(S)) = \pi_{D(k)}^k(S)$, and the cutset inequality associated to S is a metric inequality. □

Example 2 Consider the network depicted in Fig. 2, with four nodes (1, 2, 3 and 4) and 3 arcs ((1, 2), (3, 4) and (4, 1)). There are two commodities, 1 and 2, with demands d_1 and d_2 , respectively, and $O(1) = 4, O(2) = 3, D(1) = D(2) = 2$.

Consider the cutset $S = \{1, 3\}$. The subnetworks $G^k = (N^k, A^k)$ for $k = 1, 2$ are shown in Fig. 3, along with the ray $(\pi(S), \alpha(S))$ associated to S . For each commod-

Fig. 2 Network for Example 2—4 nodes, 3 arcs and 2 commodities

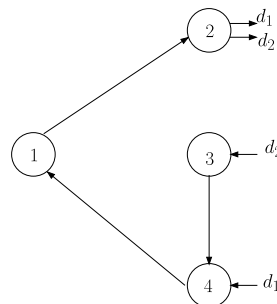
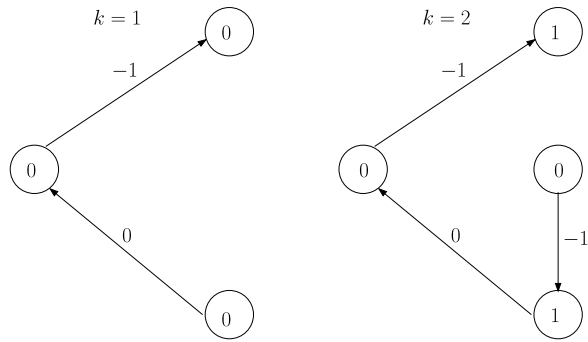


Fig. 3 Subnetworks for each commodity and ray associated to the cutset inequality for Example 2



ity k , the value inside each node i corresponds to $\pi_i^k(S)$, while the value close to each arc (i, j) is $\alpha_{ij}(S)$. This ray is associated to the Benders inequality $\omega_{12} + \omega_{34} \geq d_2$.

The length of the shortest path (with respect to $\alpha(S)$) between $O(2)$ and $D(2)$ is equal to 2. Moreover, even though neither $O(1)$ nor $D(1)$ are in set S , the shortest path length between $O(1)$ and $D(1)$ is equal to 1. Therefore, strenghtening the cutset inequality yields the metric inequality $\omega_{12} + \omega_{34} \geq d_1 + 2d_2$, which dominates the original cutset inequality.

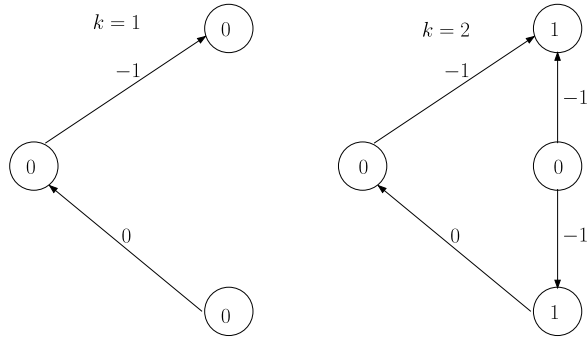
The intuition behind this strenghtening is very simple. Observe that in Example 2 the only path between $O(2)$ ($\in S$) and $D(2)$ ($\in \bar{S}$) must use more than one arc from the cutset (in this case, arcs $(3, 4)$ and $(1, 2)$), because it needs to reenter set S . Therefore, $\pi_{D(2)}^2(\alpha(S))$ actually represents the number of times an arc crossing the cutset has to be used in order for the commodity to leave its origin $O(2)$ and reach its destination $D(2)$. The cutset inequality does not take this information into account (as if a single use of the cutset arcs would always be enough) and therefore presents a worse estimation of the needed capacity. This strenghtening also considers commodities for which neither the origin nor the destination is in set S , but still need to use some capacity across the cutset, as is the case for commodity 1.

A final remark on Example 2: it is interesting to note that the same metric inequality can be obtained as the sum of two cutset inequalities: one associated with $S = \{3\}$, which yields the inequality $\omega_{34} \geq d_2$, and the other associated to $S = \{1, 3, 4\}$, which yields the inequality $\omega_{12} \geq d_1 + d_2$. There are cases, however, in which one cannot obtain the strenghtened metric inequality by adding up cutset inequalities, as shown in the next example.

Example 3 Consider the network of Example 2 with the addition of arc $(3, 2)$. Again, we consider the cutset defined by $S = \{1, 3\}$. The subnetworks for the two commodities and the ray associated to this cutset inequality are shown in Fig. 4. The cutset inequality is $\omega_{12} + \omega_{34} + \omega_{32} \geq d_2$. The strenghtened metric inequality is $\omega_{12} + \omega_{34} + \omega_{32} \geq d_1 + d_2$, which cannot be obtained as the sum of other cutset inequalities. Note, however, that it is dominated by the cutset inequality associated to $S = \{1, 3, 4\}$, which is $\omega_{12} + \omega_{32} \geq d_1 + d_2$.

The last two examples show that strenghtened metric inequalities can be dominated by other cutset inequalities. In spite of this fact, strenghtening cutset inequalities is interesting, because solution algorithms usually make use of heuristic methods

Fig. 4 Subnetworks for each commodity and ray associated to the cutset inequality for Example 3



to generate cutset inequalities. Hence, they do not guarantee that the best cutsets are generated. The proposed strengthening can be useful in that context, as confirmed by our computational experiments, reported in Sect. 4.1.

4 Computational results

To perform our computational experiments, we use instances of the *multicommodity capacitated fixed-charge network design problem* (MCFND), for which several exact and approximate solution algorithms have been proposed [7, 8, 10, 11, 13, 14, 16, 17, 20]. To each arc $(i, j) \in A$ is associated a binary variable y_{ij} taking value 1 and only if the arc is chosen, and continuous variables $x_{ij}^k, k \in K$, representing the flow of commodity k on the arc. Parameters f_{ij} and c_{ij}^k are, respectively, the fixed cost for opening arc (i, j) and the variable cost associated with the flow of one unit of commodity k through arc (i, j) , while u_{ij} is the capacity of arc (i, j) when it is open. With this notation, one can write the MCFND as follows:

$$\text{Minimize } \sum_{(i,j) \in A} \left(f_{ij} y_{ij} + \sum_{k \in K} c_{ij}^k x_{ij}^k \right) \tag{10}$$

subject to

$$\sum_{j \in N_i^+} x_{ij}^k - \sum_{j \in N_i^-} x_{ji}^k = \begin{cases} d_k, & i = O(k), \\ 0, & i \notin \{O(k), D(k)\}, \\ -d_k, & i = D(k), \end{cases} \quad \forall i \in N, \forall k \in K, \tag{11}$$

$$\sum_{k \in K} x_{ij}^k \leq u_{ij} y_{ij}, \quad \forall (i, j) \in A, \tag{12}$$

$$x_{ij}^k \geq 0, \quad \forall (i, j) \in A, \forall k \in K, \tag{13}$$

$$y_{ij} \in \{0, 1\}, \quad \forall (i, j) \in A. \tag{14}$$

Our computational experiments are based on three sets of instances widely used in the literature on the MCFND. These instances contain from 10 to 100 nodes,

35 to 687 arcs and 10 to 400 commodities. They are described in detail by Ghamlouche, Crainic and Gendreau [14].

In the next subsection, we present results obtained on cutset inequalities generated by an enumeration of all cutsets, up to a limited cardinality. We are interested in the proportion of cutset inequalities that are strengthened, as well as by the improvement obtained by the strengthened metric inequalities. Subsequently, in Sect. 4.2, we analyze the results obtained by a Benders algorithm applied to the MCFND. We investigate, in particular, the relative performance of three variants of the algorithm: one that generates Benders inequalities associated to rays that are not necessarily extreme; a second variant that generates only Benders inequalities associated to extreme rays; and a third variant that generates Benders inequalities and strengthens them to obtain only metric inequalities.

4.1 Cutset inequalities

In the special case of cutset inequalities, we have considered for each instance all sets S , with $|S| \leq \Psi$, Ψ being an integer which depends on the number of nodes in the graph (see Tables 1 and 2). All the associated cutsets have been generated and tested. In Table 1 the results are shown for the first group of instances (group R), while Table 2 shows the results obtained for the third group (group C+). For the second group (group C), considering sets with $|S| \leq 4$, no cutset inequality has been strengthened. In the tables, for each instance, we show the following

Table 1 Effect of strengthening on cutset inequalities (group R)

Inst.	(N , A , K)	Ψ	Nb. cutsets	Nb. strength	Max strength (%)	Avg strength (%)
R01.1	(10,35,10)	5	637	32	42.11	32.35
R02.1	(10,35,25)	5	637	76	34.97	19.41
R03.1	(10,35,50)	5	637	63	24.02	10.51
R04.1	(10,60,10)	5	637	0	—	—
R05.1	(10,60,25)	5	637	0	—	—
R06.1	(10,60,50)	5	637	0	—	—
R07.1	(10,82,80)	5	637	0	—	—
R08.1	(10,83,25)	5	637	0	—	—
R09.1	(10,83,50)	5	637	0	—	—
R10.1	(20,120,40)	5	6195	2	14.70	14.18
R11.1	(20,120,100)	5	6195	48	12.26	4.28
R12.1	(20,120,200)	5	6195	67	4.98	2.92
R13.1	(20,220,40)	5	6195	0	—	—
R14.1	(20,220,100)	5	6195	0	—	—
R15.1	(20,220,200)	5	6195	0	—	—
R16.1	(20,314,40)	5	6195	0	—	—
R17.1	(20,318,100)	5	6195	0	—	—
R18.1	(20,315,200)	5	6195	0	—	—

Table 2 Effect of strengthening on cutset inequalities (group C+)

Inst.	(N , A , K)	Ψ	Nb. cutsets	Nb. strength	Max strength (%)	Avg strength (%)
c25_100_10_F_L_5	(25,100,10)	4	15.275	22	100.00	45.71
c25_100_10_F_T_5	(25,100,10)	4	15.275	41	100.00	88.09
c25_100_10_V_L_5	(25,100,10)	4	15.275	160	100.00	65.77
c25_100_30_F_L_5	(25,100,30)	4	15.275	143	50.00	18.45
c25_100_30_F_T_5	(25,100,30)	4	15.275	868	100.00	25.07
c25_100_30_V_T_5	(25,100,30)	4	15.275	785	100.00	29.80
c100_400_10_F_L_10	(100,400,10)	3	166.750	3	100.00	84.84
c100_400_10_F_T_10	(100,400,10)	3	166.750	6	100.00	60.73
c100_400_10_V_L_10	(100,400,10)	3	166.750	98	100.00	56.00
c100_400_30_F_L_10	(100,400,30)	3	166.750	595	100.00	63.47
c100_400_30_F_T_10	(100,400,30)	3	166.750	304	100.00	75.91
c100_400_30_V_T_10	(100,400,30)	3	166.750	592	100.00	63.18

information:

- $(|N|, |A|, |K|)$ —number of nodes, arcs and commodities in the instance;
- Ψ —maximum cardinality of the tested cutsets;
- Nb. cutsets—number of cutsets tested;
- Nb. strength—number of cutset inequalities that were strengthened;
- Max strength—the maximum strength obtained;
- Avg strength—the average strength considering all the strengthened inequalities;

where the *strength* is defined as $(\sum_{k \in K} d^k \pi_{D(k)}^k(\alpha) - \sum_{k \in K} d_k \pi_{D(k)}^k) / \sum_{k \in K} d^k \pi_{D(k)}^k(\alpha)$.

The results presented in Tables 1 and 2 indicate that the strengthening is only effective for sparse graphs. This result was expected, since the higher the arc density, the smaller the probability that the conditions given in Proposition 4 are satisfied. Indeed, consider a subset S of the nodes of a complete graph, and its complement \bar{S} . If some commodity has its origin in S and its destination in \bar{S} , it is always possible to connect the origin to the destination by using a single arc across the cutset. Similarly, if we consider a commodity with its origin in \bar{S} or its destination in S , we can reach the destination from the origin by an arc that does not cross the cutset. Thus, in the case of a complete graph, strengthening cutset inequalities is useless, since they are already metric inequalities.

4.2 Benders inequalities

We have developed a standard Benders decomposition algorithm which obtains values for the dual variables (α, π) by using a state-of-the-art linear programming software package (CPLEX) to solve the multicommodity flow subproblem at each iteration of the algorithm.

The Benders algorithm first relaxes all feasibility and optimality constraints in the master problem. At each iteration, the relaxed master problem provides a lower bound on the optimal solution value of the original problem and a solution \bar{y} . These variables define a tentative network, and are used in the dual subproblem.

In case the dual subproblem is unbounded, we obtain a ray (not necessarily extreme) that can be used to generate a violated Benders inequality (also called a feasibility cut). In case the dual subproblem is bounded, the conjunction of the master problem and subproblem solutions is a feasible solution to the original problem (and provides an upper bound). In this case, the extreme point corresponding to a dual optimal solution can be used to generate a so-called optimality cut. The process iterates until the values of the lower and upper bounds coincide.

We have analyzed the number of iterations and the time it takes for this Benders decomposition algorithm to find a first feasible solution. We compare these two figures with and without strengthening the feasibility cuts to obtain metric inequalities. The number of iterations performed before a first feasible solution is found gives a measure of how much we can strengthen the cuts. The comparison of the computation time between the two cases is useful to determine if the time spent in strengthening is rewarded by the gain in the strength of the cuts.

In Tables 3–7, the following results are shown:

- With or Without strength—indicates the use (or not) of strengthening:
 - Nb. iter—number of iterations needed to find a first feasible solution;
 - Time (s)—time in seconds needed to find a first feasible solution;
- \neq (%)—percentage difference between the equivalent columns with and without strengthening;
- Strength info—information associated with strengthening:
 - Nb. strength—number of iterations where the cut was strengthened;
 - Max strength—maximum strength obtained (%);
 - Avg strength—average strength obtained (%).

Tables 3 and 4 show the results for the first group of instances. We see from these tables that strengthening the feasibility cuts not only reduces the number of iterations needed to find a feasible solution for most instances, but also the time needed, indicating that the time spent to solve the shortest path problem is rewarded by the strengthening of the cuts. Note that, for each line in the tables, there is actually a set of instances corresponding to the same network, but having different configurations of fixed and variable costs [14]. In the tables, we present the average results.

The results for the second group of instances are presented in Tables 5 and 6. These instances are harder than the first group. Indeed, for about half of these instances (those in Table 6), the Benders algorithm could not find a single feasible solution in the allowed computation time of five hours. Therefore, Table 6 does not compare the difference between the total time to find a feasible solution, but the lower bound upon termination. Although it is a different measure, the lower bounds are also relevant since they somehow indicate the strength of the cuts.

For the instances in Table 5, a mean reduction of 22% in the computation time needed to obtain the first feasible solution was obtained by using strengthening. The results are consistent with those of the first group of instances and indicate that the harder the problem, the more effective the strengthening seems to be in reducing the total computation time.

An interesting result is obtained in Table 6. In this case, the time limit of 5 hours was not enough to obtain a single feasible solution. However, one observes that the lower bounds obtained by the Benders decomposition algorithm using strengthening were slightly better (mean increase of 4.13%) than those obtained without strengthening. This indicates that the behavior is similar to the other two groups of instances. Finally, Table 7 shows the results for the last group of instances. Again, these results are similar to those for the other groups.

A final study concerns the relative strength of Benders inequalities and Benders inequalities associated to extreme rays. It is common in the literature to use extreme rays when solving problems by means of a Benders decomposition approach, since they are believed to yield stronger cuts. When solving an unbounded subproblem with CPLEX we had the option of obtaining an extreme ray or a ray (not necessarily extreme). We tested both options on the first group of instances and the results are summarized in Table 8.

These results somehow contradict what is commonly believed, since no clear answer can be given to the question of which is the best option. Our belief is that,

Table 3 Results for Benders inequalities (group R 10 nodes)

Instance	Without strength		With strength		≠ (%)	Strength info		Avg strength	
	Nb. iter	Time (s)	Nb. iter	Time (s)		Nb. strength	Max strength		
R01	45.50	0.10	37.67	0.09	-17.15	-9.14	11.00	28.81	16.06
R02	54.83	0.20	41.33	0.17	-24.90	-18.78	19.67	47.41	19.46
R03	58.50	0.39	44.50	0.31	-23.46	-20.49	29.00	48.79	19.41
R04	31.44	0.08	31.22	0.09	-0.60	12.53	0.33	5.64	16.91
R05	71.78	0.48	66.67	0.46	-6.61	-2.61	17.44	13.87	4.94
R06	94.56	1.49	76.78	1.19	-14.45	-16.23	33.78	20.33	6.82
R07	38.22	0.14	37.11	0.15	-2.89	4.86	1.89	7.63	6.18
R08	79.44	0.81	81.89	0.87	3.70	6.64	11.67	9.86	4.03
R09	117.22	2.79	96.22	2.02	-16.28	-25.87	36.25	25.80	5.86
Mean					-10.10	-6.40	17.25	20.87	9.61

Table 4 Results for Benders inequalities (group R 20 nodes)

Instance	Without strength		With strength		≠ (%)	Strength info		Avg strength	
	Nb. iter	Time (s)	Nb. iter	Time (s)		Nb. strength	Max strength		
R10	265.44	10.86	225.89	9.01	-12.66	-13.38	112.56	32.11	8.99
R11	282.33	48.18	204.44	34.04	-24.18	-27.83	136.44	62.92	14.57
R12	209.67	57.20	141.67	36.32	-30.02	-30.98	116.11	57.36	16.22
R13	431.78	97.43	399.89	91.72	-8.12	-6.84	117.89	17.51	3.99
R14	760.56	483.62	597.78	393.95	-17.93	-17.69	339.44	37.95	5.03
R15	542.00	708.65	353.78	483.58	-27.13	-23.70	255.22	62.34	7.83
R16	389.44	159.52	351.22	144.46	-9.60	-4.34	80.56	17.57	3.23
R17	1202.00	2315.23	983.78	1958.08	-18.05	-15.04	519.67	35.36	4.44
R18	978.89	3413.53	654.00	2471.57	-28.03	-26.11	446.22	35.34	4.86
Mean					-19.53	-18.44	236.01	39.83	7.68

Table 5 Results for Benders inequalities (group C)

Instance	Without strength		With strength		≠ (%)	Time	Strength info		Avg strength
	Nb. iter	Time (s)	Nb. iter	Time (s)			Nb. strength	Max strength	
c33	532	111.48	478	103.62	-10.15	-7.59	140	20.13	3.88
c35	454	141.42	393	120.56	-13.44	-17.30	88	16.54	3.40
c36	492	197.67	450	168.32	-8.54	-17.44	85	28.55	3.90
c37	2227	8041.07	1589	6298.42	-28.65	-27.67	1007	44.68	3.83
c38	2909	12738.5	1948	8906.8	-33.04	-43.02	1337	56.38	3.91
c39	2469	11205.6	1655	8391.61	-32.97	-33.53	999	47.60	4.06
c40	2653	12295.4	1750	9123.37	-34.04	-34.77	1105	58.41	3.17
c41	602	325.07	557	306.45	-7.48	-6.08	148	33.80	3.87
c42	588	331.64	553	282.48	-5.95	-17.40	68	15.35	3.17
c43	630	588.32	643	629.94	2.06	6.61	88	38.89	3.70
c44	911	893.24	793	658.8	-12.95	-35.59	107	13.71	3.08
c45	2959	18532.3	2179	14782	-26.36	-25.37	1367	46.06	2.65
c46	3024	18241.6	2206	14690.9	-27.05	-24.17	1300	32.49	2.60
c47	3222	21230	2193	15997.3	-31.94	-32.71	1294	45.43	2.61
c48	2772	15250.2	1991	12312.1	-28.17	-23.86	1127	52.36	2.71
Mean					-19.91	-22.66	684	36.69	3.37

Table 6 Results for Benders inequalities (group C—time limit exceeded)

Instance	Without strength		With strength		\neq (%)	Strength info		
	Nb. iter	LB	Nb. iter	LB		LB	Nb. strength	Max strength
c49	849	3677.12	658	3658.39	-0.51	637	28.51	4.65
c50	1077	16381.4	766	16341.7	-0.24	736	32.41	4.65
c51	679	2895.79	549	2930.23	1.19	515	26.01	4.13
c52	760	17175.7	555	17320.2	0.84	529	26.76	4.62
c53	1061	10920.4	560	11977	9.68	553	78.01	11.56
c54	886	20131.3	472	21416.9	6.39	461	49.34	9.25
c55	769	11105.6	436	11956.4	7.66	423	78.92	8.30
c56	741	22470.3	421	24242.4	7.89	410	54.51	8.90
c57	388	2595.49	344	2676.2	3.11	317	24.44	4.59
c58	432	5140.63	347	5233.47	1.81	316	28.91	4.68
c59	501	2512.28	381	2538.63	1.05	352	27.53	3.80
c60	287	5271.1	236	5299.48	0.54	222	21.37	3.64
c61	690	8728.18	338	9245.4	5.93	314	63.75	13.24
c62	587	17684.2	317	19029.1	7.61	306	45.13	11.41
c63	628	8467.78	325	8988.69	6.15	310	49.36	9.33
c64	492	17317.9	264	18521.3	6.95	243	40.97	10.22
Mean					4.13	415.25	42.25	7.31

although the set of extreme rays is limited (and therefore, common sense would lead us to believe that the associated Benders feasibility cuts are stronger), combining two or more Benders feasibility cuts might sometimes be more effective in cutting the feasible space.

5 Conclusion

In this paper, we clarified the relationships between three well-known classes of inequalities used in solution algorithms for multicommodity capacitated network design problems: Benders, metric and cutset inequalities. We have shown that Benders inequalities associated to extreme rays are always metric, but that it can be interesting to generate other Benders inequalities, and strengthen them to obtain metric inequalities. This has been shown in particular on cutset inequalities, a subclass of Benders inequalities, for which we have given a necessary and sufficient condition for them to be metric inequalities. Computational results on a Benders decomposition algorithm have shown that, for some instances, the time needed to find a first feasible solution was reduced by more than 30% when strengthening Benders feasibility cuts to obtain metric inequalities.

Our results are applicable not only in the context of solving the MCFND by a Benders decomposition algorithm. In particular, as mentioned in the Introduction, our theoretical results apply to any network design problem for which feasible solutions can be obtained by solving multicommodity network flow subproblems. Also,

Table 7 Results for Benders inequalities (group C+)

Instance	Without strength		With strength		≠ (%)		Strength info	
	Nb. iter	Time (s)	Nb. iter	Time (s)	Nb. iter	Time	Nb. strength	Avg strength
c100_400_10_F_L_10	897	106.42	902	105.5	0.56	-0.86	56	21.88
c100_400_10_F_T_10	2170	491.34	2083	495.85	-4.01	0.92	59	33.49
c100_400_10_V_L_10	1361	312.08	1352	254.95	-0.66	-18.31	88	22.22
c100_400_30_F_L_10	6109	10871.3	5240	9678.83	-14.22	-10.97	1620	40.20
c100_400_30_F_T_10	3374	2361.29	3027	2085	-10.28	-11.70	1070	41.78
c25_100_10_F_L_5	171	1.06	158	1.09	-7.60	2.83	31	36.84
c25_100_10_F_T_5	133	0.71	127	0.73	-4.51	2.82	14	25.68
c25_100_10_V_L_5	115	0.64	95	0.54	-17.39	-15.63	25	25.81
c25_100_30_F_L_5	300	6.21	233	5.09	-22.33	-18.04	105	45.16
c25_100_30_F_T_5	226	3.7	164	2.86	-27.43	-22.70	99	69.80
c25_100_30_V_T_5	233	4.04	182	3.22	-21.89	-20.30	118	51.31
Mean					-11.80	-10.18	298.63	37.65
								12.54

Table 8 Effect of using Benders inequalities associated to extreme rays

Instance	Benders inequalities		Benders inequalities (extreme rays)	
	Nb. iter	Time (s)	Nb. iter	Time (s)
R01	40.33	0.14	34.67	0.13
R02	44.17	0.24	40.00	0.24
R03	43.67	0.39	37.00	0.34
R04	32.44	0.15	32.89	0.17
R05	67.33	0.65	71.22	0.81
R06	72.33	1.32	64.56	1.26
R07	34.00	0.19	33.56	0.24
R08	81.89	1.15	73.56	1.15
R09	98.44	2.74	87.44	2.70
R10	225.11	11.59	202.56	14.58
R11	205.56	37.95	154.11	38.10
R12	143.22	38.00	116.89	33.45
R13	396.33	107.65	340.78	89.32
R14	581.33	441.89	694.11	782.58
R15	347.00	492.39	414.22	1760.19
R16	347.11	158.10	325.33	126.93
R17	963.44	1938.99	1413.00	5348.33
R18	638.78	2530.32	820.11	9646.16

the strengthening procedure to generate metric inequalities can be used in any algorithm (for the MCFND or for any other network design problem) that alternates between solving a relaxation, gradually improved by the addition of valid inequalities, and restricted multicommodity flow subproblems. Evaluating the performance of the strengthening procedure in this context constitutes an interesting avenue for future research.

References

1. Atamtürk, A.: On capacitated network design cut-set polyhedra. *Math. Program.* **92**, 425–437 (2002)
2. Atamtürk, A.: On splittable and unsplittable capacitated network design arc-set polyhedra. *Math. Program.* **92**, 315–333 (2002)
3. Balakrishnan, A., Magnanti, T.L., Mirchandani, P.: Network design. In: Dell’Amico, M., Maffioli, F., Martello, S. (eds.) *Annotated Bibliographies in Combinatorial Optimization*, pp. 311–334. Wiley, New York (1997)
4. Barahona, F.: Network design using cut inequalities. *SIAM J. Optim.* **6**, 823–837 (1996)
5. Benders, J.F.: Partitioning procedures for solving mixed variables programming problems. *Numer. Math.* **4**, 238–252 (1962)
6. Chouman, M., Crainic, T.G., Gendron, B.: A cutting-plane algorithm based on cutset inequalities for multicommodity capacitated fixed charge network design. Publication CRT-316, Centre de recherche sur les transports, Université de Montréal (2003)
7. Crainic, T.G., Gendreau, M., Farvolden, J.M.: A simplex-based tabu search method for capacitated network design. *INFORMS J. Comput.* **12**, 223–236 (2000)

8. Crainic, T.G., Gendron, B., Hernu, G.: A slope scaling/lagrangean perturbation heuristic with long-term memory for multicommodity fixed-charge network design. *J. Heuristics* **10**, 525–545 (2004)
9. Gabrel, V., Knippel, A., Minoux, M.: Exact solution of multicommodity network optimization problems with general step cost functions. *Oper. Res. Lett.* **25**, 15–23 (1999)
10. Gendron, B., Crainic, T.G.: Relaxations for multicommodity capacitated network design problems. Publication CRT-965, Centre de recherche sur les transports, Université de Montréal (1994)
11. Gendron, B., Crainic, T.G.: Bounding procedures for multicommodity capacitated fixed charge network design problems. Publication CRT-96-06, Centre de recherche sur les transports, Université de Montréal (1996)
12. Gendron, B., Crainic, T.G., Frangioni, A.: Multicommodity capacitated network design. In: Sanso, B., Soriano, P. (eds.) *Telecommunications Network Planning*, pp. 1–19. Kluwer, Norwell (1999)
13. Ghamlouche, I., Crainic, T.G., Gendreau, M.: Cycle-based neighbourhoods for fixed-charge capacitated multicommodity network design. *Oper. Res.* **51**, 655–667 (2003)
14. Ghamlouche, I., Crainic, T.G., Gendreau, M.: Path relinking, cycle-based neighbourhoods and capacitated multicommodity network design. *Ann. Oper. Res.* **131**, 109–133 (2004)
15. Günlük, O.: A branch-and-cut algorithm for capacitated network design problems. *Math. Program.* **86**, 17–39 (1999)
16. Holmberg, K., Yuan, D.: A lagrangean heuristic based branch-and-bound approach for the capacitated network design problem. *Oper. Res.* **48**, 461–481 (2000)
17. Klierer, G., Timajev, L.: Relax-and-cut for capacitated network design. In: *Proceedings of the 13th Annual European Symposium on Algorithms (ESA)*. Lecture Notes on Computer Science, vol. 3669, pp. 47–58. Springer, Berlin (2005)
18. Magnanti, T.L., Wong, R.T.: Network design and transportation planning: models and algorithms. *Transp. Sci.* **18**, 1–56 (1984)
19. Minoux, M.: Network synthesis and optimum network design problems: models, solution methods and applications. *Networks* **19**, 313–360 (1989)
20. Sellman, M., Klierer, G., Koberstein, A.: Lagrangian cardinality cuts and variable fixing for capacitated network design. In: *Proceedings of the 10th Annual European Symposium on Algorithms (ESA)*. Lecture Notes on Computer Science, vol. 2461, pp. 845–858. Springer, Berlin (2002)
21. Stoer, M., Dahl, G.: A polyhedral approach to multicommodity survivable network design. *Numer. Math.* **68**, 149–167 (1994)